

Math7501 Examination 2010: Sample Solutions and Marking Schemes

1. (a) Let Ω and \mathcal{F} respectively denote the sample space and the event space. The three axioms of probability for a probability function $P(\cdot)$ are:

- (i) For any $E \in \mathcal{F}$, $P(E) \geq 0$.
- (ii) $P(\Omega) = 1$.
- (iii) If $E_1 \in \mathcal{F}, E_2 \in \mathcal{F}, \dots$ are mutually disjoint, then $P\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} P(E_j)$.

Now

$$P(A \cup B) = P(A \cup (B \cap A^c)) = P(A) + P(B \cap A^c) \quad (\text{axiom (iii)}).$$

But $P(B) = P(B \cap A) + P(B \cap A^c)$ (axiom (iii)) implies that

$$P(B \cap A^c) = P(B) - P(B \cap A).$$

Hence $P(A \cup B) = P(A) + P(B) - P(B \cap A)$ as required.

- (b) Events A , B and C are mutually independent if $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(B \cap C) = P(B)P(C)$, and $P(A \cap B \cap C) = P(A)P(B)P(C)$.

Now

$$\begin{aligned} & P(A \cap (B \cup C)) \\ &= P((A \cap B) \cup (A \cap C)) \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \quad (\text{from axioms}) \\ &= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) \quad (\text{by assumption}) \\ &= P(A)[P(B) + P(C) - P(B)P(C)] \\ &= P(A)P(B \cup C) \end{aligned}$$

(c) $P(A \cap B) = P(B|A)P(A) = 1/12$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) = 1/4 + P(B) - 1/12 \\ &= 1/6 + P(B) \quad \text{as required.} \end{aligned}$$

If A and B are independent, $P(A \cap B) = P(A)P(B)$. Now, $1/3 = P(A \cup B) = 1/6 + P(B)$, so $P(B) = 1/6$. Then $P(A)P(B) = 1/4 \times 1/6 = 1/24 \neq 1/12 = P(A \cap B)$, so A and B are not independent.

2. Let A and B respectively denote the events that the shop is supplied by factory A and factory B. Let D denote the event that a component is defective. Then $P(A) = P(B) = 1/2$, $P(D|A) = 1/50$, $P(D|B) = 1/200$. Let D_j be the event that the j th tested component is defective.

(a) For the first tested component,

$$\begin{aligned} P(D_1) &= P(D_1|A)P(A) + P(D_1|B)P(B) = \frac{1}{2} \times \frac{1}{50} + \frac{1}{2} \times \frac{1}{200} = \frac{1}{80} \\ P(A|D_1) &= \frac{P(D_1|A)P(A)}{P(D_1)} = \frac{\frac{1}{2} \times \frac{1}{50}}{\frac{1}{80}} = \frac{4}{5} \end{aligned}$$

(b) Given the first component defective, we have updated $P(A|D_1) = 4/5$ and $P(B|D_1) = 1/5$. Then for the second component, by definition of conditional probability and Law of Total Probability,

$$\begin{aligned} P(D_2|D_1) &= P(D_2|D_1 \cap A)P(A|D_1) + P(D_2|D_1 \cap B)P(B|D_1) \\ &= \frac{1}{50} \times \frac{4}{5} + \frac{1}{200} \times \frac{1}{5} = \frac{17}{1000} \end{aligned}$$

or,

$$\begin{aligned} P(D_2|D_1) &= \frac{P(D_1 \cap D_2)}{P(D_1)} \\ &= \frac{P(D_1 \cap D_2|A)P(A) + P(D_1 \cap D_2|B)P(B)}{P(D_1)} \\ &= \frac{(1/2500) \times (1/2) + (1/40000) \times (1/2)}{1/80} = \frac{17}{1000} \end{aligned}$$

Given that the second component is not defective, we have

$$\begin{aligned} P(A|D_1 \cap D_2^c) &= \frac{P(D_1 \cap D_2^c|A)P(A)}{P(D_1 \cap D_2^c|A)P(A) + P(D_1 \cap D_2^c|B)P(B)} \\ &= \frac{(1/50) \times (49/50) \times (1/2)}{(1/50) \times (49/50) \times (1/2) + (1/200) \times (199/200) \times (1/2)} \\ &= \frac{784}{983} = 0.7976 \end{aligned}$$

or,

$$\begin{aligned} P(A|D_1 \cap D_2^c) &= \frac{P(D_2^c|A \cap D_1)P(A|D_1)}{P(D_2^c|A \cap D_1)P(A|D_1) + P(D_2|B \cap D_1)P(B|D_1)} \\ &= \frac{(49/50) \times (4/5)}{(49/50) \times (4/5) + (199/200) \times (1/5)} = \frac{784}{983} = 0.7976 \end{aligned}$$

(c) Let N_k be the event that k out of n are defective.

$$\begin{aligned} P(N_k) &= P(N_k|A)P(A) + P(N_k|B)P(B) \\ &= \frac{1}{2} \left[\binom{n}{k} (0.02)^k (0.98)^{n-k} + \binom{n}{k} (0.005)^k (0.995)^{n-k} \right] \end{aligned}$$

$$\begin{aligned} P(A|N_k) &= \frac{P(N_k|A)P(A)}{P(N_k)} = \frac{\frac{1}{2} \binom{n}{k} (0.02)^k (0.98)^{n-k}}{\frac{1}{2} \left[\binom{n}{k} (0.02)^k (0.98)^{n-k} + \binom{n}{k} (0.005)^k (0.995)^{n-k} \right]} \\ &= \frac{20^k 980^{n-k}}{20^k 980^{n-k} + 5^k 995^{n-k}} \end{aligned}$$

3. (a) For $p_X(x)$ to be a pmf, we require that $\sum_{x=1}^{\infty} p_X(x) = 1$, i.e.

$$k \sum_{x=1}^{\infty} \frac{p^x}{x} = 1, \quad k(-\ln(1-p)) = 1, \quad k = \frac{-1}{\ln q}$$

$q = 1 - p < 1$, so $\ln q < 0$ and k is positive.

(b) The probability generating function of X is

$$\Pi_X(t) = E(t^X) = \frac{1}{\ln q} \sum_{x=1}^{\infty} t^x \frac{-p^x}{x} = \frac{1}{\ln q} \sum_{x=1}^{\infty} \frac{-(pt)^x}{x} = \frac{\ln(1-pt)}{\ln q}, \quad |t| < 1/p$$

If $|t| < 1/p$, then $|pt| < 1$ and $\ln(1-pt)$ is well defined.

(c)

$$E(X) = \frac{d}{dt} \Pi_X(t) \Big|_{t=1} = \frac{d}{dt} \frac{\ln(1-pt)}{\ln q} \Big|_{t=1} = \frac{-p}{1-pt} \frac{1}{\ln q} \Big|_{t=1} = \frac{-p}{q \ln q}$$

or

$$E(X) = \sum_{x=1}^{\infty} \frac{-p^x}{\ln q} = \frac{-1}{\ln q} \sum_{x=1}^{\infty} p^x = \frac{-1}{\ln q} \frac{p}{1-p} = \frac{-p}{q \ln q}$$

(d) For $Var(X)$,

$$\begin{aligned} E(X(X-1)) &= \frac{d^2}{dt^2} \Pi_X(t) \Big|_{t=1} = \frac{d^2}{dt^2} \frac{\ln(1-pt)}{\ln q} \Big|_{t=1} \\ &= \frac{d}{dt} \frac{-p}{1-pt} \frac{1}{\ln q} \Big|_{t=1} = -\frac{p^2}{(1-pt)^2} \frac{1}{\ln q} \Big|_{t=1} = \frac{-p^2}{q^2 \ln q}. \end{aligned}$$

$$\begin{aligned} Var(X) &= E(X(X-1)) + E(X) - (E(X))^2 \\ &= \frac{-p^2}{q^2 \ln q} + \frac{-p}{q \ln q} - \frac{p^2}{q^2} \frac{1}{(\ln q)^2} = \frac{-p(p + \ln q)}{q^2 (\ln q)^2} \end{aligned}$$

4. (a) (i) This is a gamma distribution with shape parameter α and scale parameter β . The mean is α/β and the variance is α/β^2 .
- (ii) The MGF of X is

$$\begin{aligned}
 M_X(t) &= E(e^{tX}) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta-t)x} dx \\
 &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{u}{\beta-t} \right)^{\alpha-1} e^{-u} \frac{du}{\beta-t} \quad (u = (\beta-t)x) \\
 &= \left(\frac{\beta}{\beta-t} \right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} du = \left(1 - \frac{t}{\beta} \right)^{-\alpha}
 \end{aligned}$$

- (b) (i) The MGF of S is

$$\begin{aligned}
 M_S(t) &= E(e^{tS}) = E\left(\prod_{i=1}^n e^{tX_i}\right) = \prod_{i=1}^n M_{X_i}(t) \\
 &= \prod_{i=1}^n \left(1 - \frac{t}{\beta}\right)^{-\alpha} = \left(1 - \frac{t}{\beta}\right)^{-n\alpha}
 \end{aligned}$$

which is the MGF of a gamma distribution with parameters $n\alpha$ and β . Therefore, S has a gamma distribution with shape parameter $n\alpha$ and scale parameter β .

- (ii) By CLT, the distribution of S is approximately $N(90 \cdot 10 / 0.3, 90 \cdot 10 / 0.3^2)$. Hence

$$P(S > 3100) = P\left(\frac{S - 3000}{\sqrt{900/0.3^2}} > \frac{3100 - 3000}{\sqrt{900/0.3^2}}\right) \approx P(Z > 1) = 0.1587$$

where $Z \sim N(0, 1)$.

5. (a) (i) $b(T, \theta) = E(T) - \theta$, $mse(T) = E[(T - \theta)^2]$.

(ii)

$$\begin{aligned} mse(T) &= E[(T - E(T) + E(T) - \theta)^2] \\ &= E[(T - E(T))^2 + (E(T) - \theta)^2 + 2(T - E(T))(E(T) - \theta)] \\ &= E[(T - E(T))^2 + (E(T) - \theta)^2] \end{aligned}$$

Note that $E[(T - E(T))(E(T) - \theta)] = [E(T) - \theta]E[T - E(T)] = 0$. Hence,

$$mse(T) = Var(T) + b^2(T, \theta).$$

(b) (i) $Y \sim Bin(n, p)$, so $E(Y) = np$, $Var(Y) = np(1-p)$.

(ii) $T = n \frac{Y}{n} \left(1 - \frac{Y}{n}\right)$, so

$$\begin{aligned} E(T) &= E(Y) - \frac{1}{n}E(Y^2) = np - \frac{1}{n}(np(1-p) + n^2p^2) \\ &= np - (n-1)p^2 - p = (n-1)p(1-p) \\ &\neq np(1-p) = Var(Y) \end{aligned}$$

Hence T is a biased estimator of $Var(Y) = np(1-p)$. Let $W = n(n-1)^{-1}T$, then $E(W) = \frac{n}{(n-1)}E(T) = \frac{n}{(n-1)}(n-1)p(1-p) = np(1-p)$. Hence

$$W = \frac{n}{n-1}T = \frac{n}{n-1}n \frac{Y}{n} \left(1 - \frac{Y}{n}\right) = \frac{nY}{n-1} \left(1 - \frac{Y}{n}\right)$$

is an unbiased estimator of $Var(Y)$.

(c) (i) $Y \sim Poi(\lambda)$, so $E(Y) = \lambda$ and $Var(Y) = \lambda$. Hence

$$\begin{aligned} E(C) &= E(5Y + Y^2) = 5E(Y) + E(Y^2) = 5E(Y) + Var(Y) + (E(Y))^2 \\ &= 5\lambda + \lambda + \lambda^2 = 6\lambda + \lambda^2. \end{aligned}$$

(ii)

$$E(5Y_1 + Y_1^2) = 5E(Y_1) + E(Y_1^2) = 5\lambda + \lambda + \lambda^2 = 6\lambda + \lambda^2.$$

$$\begin{aligned} E\left(\frac{5}{n} \sum_{i=1}^n Y_i + \frac{1}{n} \sum_{i=1}^n Y_i^2\right) &= \frac{5}{n} \sum_{i=1}^n E(Y_i) + \frac{1}{n} \sum_{i=1}^n E(Y_i^2) \\ &= \frac{5}{n}n\lambda + \frac{1}{n}n(\lambda^2 + \lambda) \\ &= 5\lambda + \lambda + \lambda^2 = 6\lambda + \lambda^2. \end{aligned}$$

Hence $5Y_1 + Y_1^2$ and $\frac{5}{n} \sum_{i=1}^n Y_i + \frac{1}{n} \sum_{i=1}^n Y_i^2$ are both unbiased estimators for $E(C)$. The latter has a smaller variance since it is the sample mean of the random sample $5Y_1 + Y_1^2, 5Y_2 + Y_2^2, \dots, 5Y_n + Y_n^2$.

6. (a)

$$\frac{14}{\sigma_1^2} S_1^2 \sim \chi_{14}^2, \quad \frac{14}{\sigma_2^2} S_2^2 \sim \chi_{14}^2.$$

If $\sigma_1^2 = \sigma_2^2$, then

$$\frac{14}{\sigma^2} (S_1^2 + S_2^2) \sim \chi_{28}^2.$$

(b) For testing $H_0 : \sigma_1^2 = \sigma_2^2$ versus $H_1 : \sigma_1^2 \neq \sigma_2^2$, use the test statistic

$$F = \frac{S_1^2/(15-1)}{S_2^2/(15-1)} = \frac{21}{15} = 1.4$$

H_0 would not be rejected, at the 5% level, for all values of F between the lower and upper 2.5% of a $F_{14,14}$ distribution, i.e. for all values in the range $(1/2.97, 2.97) = (0.34, 2.97)$. Since the observed value of F is 1.4, do not reject $H_0 : \sigma_1^2 = \sigma_2^2$ at the 5% level.

(c) $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$ with $\sigma_1^2 = \sigma_2^2$. The pooled sample variance is

$$S_p^2 = \frac{(15-1)S_1^2 + (15-1)S_2^2}{15+15-2} = \frac{21+15}{2} = 18$$

the test statistic is

$$T = \frac{\bar{X} - \bar{Y}}{S_p \sqrt{1/n_1 + 1/n_2}} = \frac{85 - 78}{\sqrt{18(1/15 + 1/15)}} = 4.52.$$

$H_0 : \mu_1 = \mu_2$ would not be rejected, at the 5% level, for all values of T between the lower and upper 2.5% of a t distribution with 28 degrees of freedom, i.e. for all values in the range $(-2.048, 2.048)$. Since the observed value of T is 4.52 and falls outside the the range $(-2.048, 2.048)$, reject H_0 at the 5% level.

A 95% confidence interval for $\mu_1 - \mu_2$ is

$$(85 - 78) \pm 2.048 \sqrt{18(1/15 + 1/15)} = (3.827, 10.173),$$

which does not include zero.